Lifts of Borel actions on quotient spaces
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Countable Borel equivalence relations

A **Borel equivalence relation** is an equivalence relation E on a Polish space X which is Borel as a subset of $E \subseteq X^2$.

A **countable** Borel equivalence relation (CBER) is one where every class is countable.

Every CBER comes from the following example:

Let Γ be a countable group, acting continuously on a Polish space X.

Denote by E_Γ^X the **orbit equivalence relation** of the action $\Gamma \curvearrowright X$, where x and y are related iff they are in the same orbit.

To remove uninteresting edge cases, we will only look at **aperiodic** CBERs, that is, only those where every class is infinite.

Automorphisms of CBERs

Let E be an aperiodic CBER on a Polish space X.
A Borel map $\phi : X \to X$ is an **automorphism** of E if we have

$$x \mathrel{E} y \iff \phi(x) \mathrel{E} \phi(y)$$

for every $x, y \in X$.
We denote by $\mathrm{Aut}_B(E)$ the automorphism group of E.
Every $\phi \in \mathrm{Aut}_B(E)$ descends to a Borel permutation $\widetilde{\phi}$ of X/E, defined by

$$\widetilde{\phi}([x]_E) = [\phi(x)]_E.$$

(A function $X/E \to Y/F$ is **Borel** if it lifts to a Borel map $X \to Y$ or equivalently, if the set $\{(x, y) \subseteq X \times Y : f([x]_E) = [y]_F\}$ is a Borel subset of $X \times Y$).

The inverse problem

Does every Borel permutation of X/E come from an automorphism of E?
No! Because Borel bireducibility is weaker than Borel isomorphism:
Let $E = E_0 \oplus E_t$ on $2^\omega \oplus 2^\omega$, and let f be any Borel bireduction from E_0 to E_t.
Then f induces a Borel permutation of $(2^\omega \oplus 2^\omega)/E$, but it can't lift to an automorphism of E, since this would give a Borel isomorphism from E_0 to E_t.
If E is compressible, this issue doesn't arise.
But there are probably non-compressible examples too.
If there is a free p.m.p. action $\Gamma \curvearrowright (X, \mu)$ satisfying some conditions(e.g. Γ is cohopfian, the action is cocycle superrigid, etc.)then every Borel permutation on X/Γ lifts.

More general setup: Group actions

Let E be an aperiodic CBER.
Let Γ be a countable group.

Problem

When can a Borel action of $\Gamma \curvearrowright X/E$ be lifted to a Borel action $\Gamma \curvearrowright X$?

(An action $\Gamma \curvearrowright X/E$ is **Borel** if it acts by Borel permutations of X/E)
Let $E^{\vee\Gamma}$ be the CBER generated by E and Γ:

$$x \, E^{\vee\Gamma} \, y \iff \exists\gamma(\gamma \cdot [x]_E = [y]_E)$$

We need an $(E, E^{\vee G})$-**link**.

Links

Let (E, F) be a pair of CBERs with $E \subseteq F$.
An **(E, F)**-link is a CBER $L \subseteq F$ such that on each F-class, every L-class meets every E-class exactly once.

Proposition

Let $\Gamma \curvearrowright X/E$ be a Borel action. If there is an $(E, E^{\vee\Gamma})$-link, then the action $\Gamma \curvearrowright X/E$ lifts.

If L is the link, let $\phi(x)$ be the unique element of $[x]_L \cap f([x]_E)$.

Compressible CBERs

Theorem (FKS)

Let (E, F) be compressible CBERs. Then there is an (E, F)-link.

Corollary

Let E be a compressible CBER. Then every action $\Gamma \curvearrowright X/E$ can be lifted to an action $\Gamma \curvearrowright X$.

In particular, we can always lift on a comeager set:

Corollary (FKS)

For every aperiodic CBER E and every action $\Gamma \curvearrowright X/E$, there is a comeager $E^{\vee\Gamma}$-invariant subset $Y \subseteq X$ on which the action $\Gamma \curvearrowright Y/E$ lifts.

Outer automorphisms

Let $\mathrm{Out}_B(E)$ denote the group of Borel permutations of X/E which lift to an automorphism of E.
An alternative construction of $\mathrm{Out}_B(E)$: $\mathrm{Out}_B(E)$ is the quotient $\mathrm{Aut}_B(E)/\mathrm{Inn}_B(E)$, where $\mathrm{Inn}_B(E)$ is the **inner automorphism group** of E.(Also called the **full group** of E and denoted $[E]$).

$$1 \to \mathrm{Inn}_B(E) \to \mathrm{Aut}_B(E) \to \mathrm{Out}_B(E) \to 1$$

An automorphism ϕ of E is **inner** if it satisfies $x \, E \, \phi(x)$ for every x.
So $\mathrm{Out}_B(E)$ is called the **outer automorphism group**.

Lifting outer actions

In general, there are actions on X/E which cannot be lifted (in which case E is not compressible).
Restrict to **outer actions** $\Gamma \curvearrowright X/E$, i.e. actions by elements of $\mathrm{Out}_B(E)$, i.e. homomorphisms $\Gamma \to \mathrm{Out}_B(E)$.

Problem

When can we solve the following lifting problem?

$$\begin{array}{ccc} & & \mathrm{Aut}_B(E) \\ & \nearrow & \downarrow \\ \Gamma & \longrightarrow & \mathrm{Out}_B(E) \end{array}$$

An example

Let Γ be a countable group, and let $N \triangleleft \Gamma$.
Let $\Gamma \curvearrowright X$ be a Borel action, and let E be the equivalence relation induced by the restriction $N \curvearrowright X$.
Then there is an outer action $\Gamma \to \mathrm{Out}_B(E)$ given by

$$g \cdot [x]_E = [g \cdot x]_E.$$

This descends to a outer action $\Gamma/N \to \mathrm{Out}_B(E)$.

Some groups where every outer action lifts

A tautology: Every outer action $\mathbb{Z} \to \mathrm{Out}_B(E)$ lifts (for any CBER E).
More generally, every outer action $F_n \to \mathrm{Out}_B(E)$ lifts.
We shift our focus to the class of (countable) groups for which every outer action lifts.

Finite groups

Theorem (FKS)

Let Γ be a finite group. Then every outer action of Γ lifts.

Theorem (FKS)

Let $E \triangleleft F$ be a finite index extension of CBERs. Then there is an (E, F)-link.

We say $E \triangleleft F$ (E is a **normal** subequivalence relation of F) if $F = E^{\vee\Gamma}$ for some outer action $\Gamma \to \mathrm{Out}_B(E)$.

Use **maximal fsr**.

Amalgamated products of finite groups

Lift $\Gamma *_\Lambda \Delta \to \mathrm{Out}_B(E)$.

Attempt:

1. Lift the Λ-action.
2. Then lift the Γ-action preserving the Λ-action, and similarly for Δ.

How to do the second step? Use links:

Theorem

Let $E \subseteq F \subseteq F'$ be CBERs such that E has finite index in F' and $E \lhd F'$. Then every (E, F)-link extends to an (E, F')-link.

This uses the cancellation law for cardinal algebras.

Treeability

Let $\mathcal{G}$ be the class of countable groups for which every outer action lifts.

Theorem (Frisch-Kechris-S)

Every group in $\mathcal{G}$ is treeable.

Treeable group: admits a free p.m.p. action on a standard Borel space with treeable orbit equivalence relation.
Groups which are not treeable:

- $SL_3(\mathbb{Z})$. More generally, infinite property (T) groups.
- $\mathbb{Z} \times F_2$. More generally, $\Gamma \times \Delta$ with Γ infinite and Δ non-amenable.

Problem

Is every treeable group in $\mathcal{G}$?

Proof of treeability

Theorem (Frisch-Kechris-S)

Every group in $\mathcal{G}$ is treeable.

Suppose every outer action of Γ lifts.
Write $\Gamma = F_\infty / N$ for some $N \triangleleft F_\infty$.
Fix a free p.m.p. action $F_\infty \curvearrowright (X, \mu)$, for example, the Bernoulli shift 2^{F_∞} (this is treeable).
Let E be the equivalence relation induced by $N \curvearrowright X$.
This induces an outer action $\Gamma \to \mathrm{Out}_B(E)$.
There is a lift $\Gamma \to \mathrm{Aut}_B(E)$.
This action of Γ on X is free, p.m.p. and treeable.
So Γ is treeable.

Groups which work

$\mathcal{G}$: the class of countable groups for which every outer action lifts.

Theorem (Frisch-Kechris-S)

$\mathcal{G}$ contains:

1. *all free groups;*
2. *all amenable groups.*
3. *all amalgamated products of finite groups;*

Amenable groups

Amenable group: Countable group with a Følner sequence (F_n).
Følner sequence: for every $g \in \Gamma$,

$$\frac{|F_n \triangle gF_n|}{|F_n|} \to 0$$

The F_n are "invariant as $n \to \infty$".
Some amenable groups:

- Finite groups.
- Abelian groups.
- Groups constructed from those (e.g. solvable groups).

Amenable groups: quasi-tilings

Very important tool: **quasi-tilings** (Ornstein-Weiss)
Generalized many results about actions of $\mathbb{Z}$ to amenable groups, e.g. ergodic theorem, entropy.
Idea: There's a finite set of **tiles** (finite subsets of Γ) and the group can be "disjointly" tiled with them.

Amenable groups: lifting outer actions

Strategy due to Feldman-Sutherland-Zimmer.

We want to lift an outer action $\Gamma \to \mathrm{Out}_B(E)$.

Find a set $\mathcal{A}_0$ of tiles.

Find a set $\mathcal{A}_1$ of tiles, each one quasi-tiled by $\mathcal{A}_0$.

Find a set $\mathcal{A}_2$ of tiles, each one quasi-tiled by $\mathcal{A}_1$, etc.

Define the action of Γ for elements of $\mathcal{A}_0$, then $\mathcal{A}_1$, etc.

Thank you!